UNCERTAINTY: PROBLEMS & ANSWERS

IB Physics Core: Problems 1 through 11; AHL Problems 1 through 19.

1. Four students measure the same length of string and their results are as follows:

$$\ell_1 = 38.6cm,\ \ell_2 = 38.7cm,\ \ell_3 = 38.6cm,\ \ell_4 = 38.5cm$$

What is the average or *mean* measurement?

$$\ell_{ave} = \frac{\ell_1 + \ell_2 + \ell_3 + \ell_4}{n_\ell} = \frac{38.6cm + 38.7cm + 38.6cm + 38.5cm}{4} = \frac{154.4cm}{4} = 38.6cm$$

What is the *range* of measured values?

$$R = \ell_{\max} - \ell_{\min} = 38.7cm - 38.5cm = 0.2cm$$

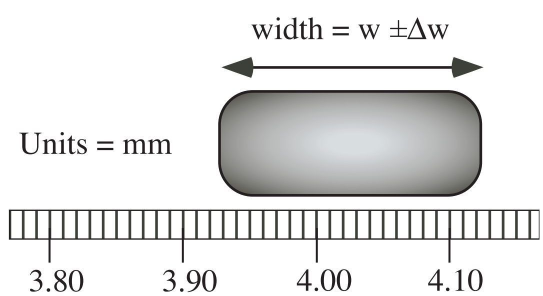
Repeated measurement of the same quantity can improve the overall acceptable value of that measurement. What is the *mean* and its *absolute uncertainty* for the length of string?

$$\Delta\ell_{ave} = \frac{R}{n} = \frac{\ell_{\max} - \ell_{\min}}{4} = \frac{0.2cm}{4} = \pm 0.05cm$$

but we have only 0.1 decimal place, hence $\Delta\ell_{ave} = \pm 0.1cm$

Therefore: $\ell \pm \Delta\ell = (38.6 \pm 0.1)cm$

2. What is the *width* and its *absolute uncertainty* of the object being measured in the sketch below?

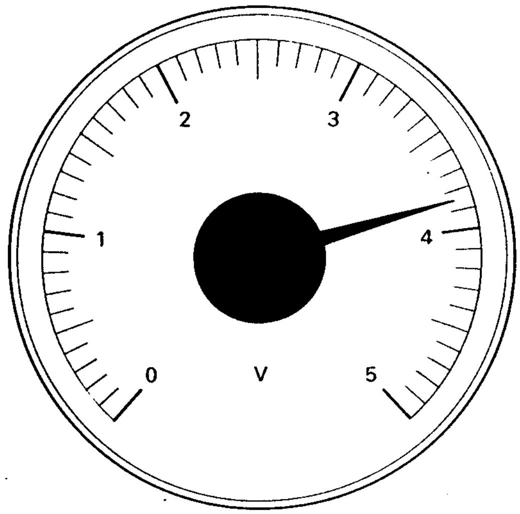


SOLUTION: $w = w_2 - w_1 = 4.12mm = 3.93mm = 0.19mm$

Smallest division is $\Delta w = 0.01mm$

$$w \pm \Delta w = 0.19mm \pm 0.01mm = (0.19 \pm 0.01)mm$$

3. The analogue voltmeter below measures a voltage on a scale of zero to 5 volts. What is the measured *voltage* and what is the *absolute uncertainty* shown here?



SOLUTION: Scale reads $V = 3.85V$

We can discern one half the smallest division so $\Delta V = \pm(0.5 \times 0.1V) = \pm 0.05V$

$$V \pm \Delta V = (3.85 \pm 0.05)V$$

4. The digital stopwatch was started at a time $t_0 = 0$ and then was used to measure ten swings of a simple pendulum to a time of $t = 17.26s$.

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If the time for ten swings of the pendulum is $17.26s$ what is the *minimum absolute uncertainty* in this measurement?

SOLUTION: $\Delta(10T) = \pm 0.01s$

What is the *time* (period T) *of one complete pendulum swing* and its *absolute uncertainty*?

SOLUTION:

$$10T = (17.26 \pm 0.01)s \quad \rightarrow \quad T \pm \Delta T = \frac{(17.26 \pm 0.01)s}{10} = (1.726 \pm 0.001)s$$

5. Add the following lengths: $12.2cm + 11.62cm + 0.891cm$

SOLUTION: $12.2cm + 11.62cm + 0.891cm = 24.711cm \approx 24.7cm$
We cannot improve precision beyond the limit of 0.1 cm.

6. Add the following distances and express your result in units of centimeters (cm).

$$1.250 \times 10^{-3} m + 25.62cm + 426mm$$

SOLUTION: $0.01250cm + 25.62cm + 42.6cm = 68.2200125cm$
When adding we cannot improve in precision,
hence we limit the sum to the 0.1cm place
Therefore the total $\approx 68.2\,cm$

7. Average speed is the ratio of distance to time: $v = \frac{s}{t}$.

That is the *average speed* if $s = 4.42\,m$ and $t = 3.085s$?

SOLUTION: $v_{ave} = \frac{s}{t} = \frac{4.42m}{3.085\,s} = 1.4327391\,m\,s^{-1}$

We only know to 3 SF, hence $v_{ave} \approx 1.43\,m\,s^{-1}$

8. Given two masses, $m_1 = (100.0 \pm 0.4)g$ and $m_2 = (49.3 \pm 0.3)g$, what is their sum, $m_1 + m_2$, and what is their difference, $m_1 - m_2$, both expressed with uncertainties.

Sum: $(149.3 \pm 0.7)\,g$ Difference: $(50.7 \pm 0.7)\,g$
We add the uncertainties in both cases.

9. With a good stopwatch and some practice, one can measure times ranging from about a second up to many minutes with an uncertainty of 0.1 second or so. Suppose that we wish to find the period τ of a pendulum with $\tau \approx 0.5s$. If we time one oscillation, we will have an uncertainty of about 20%, but by timing several successive oscillations, we can do much better.

If we measure the time for five successive oscillations and get $2.4 \pm 0.1s$, what is the final answer (with an absolute uncertainty) for the period? What if we measure 20 oscillations and get a time of $9.4 \pm 0.1s$?

For **five** oscillations: $(2.4 \pm 0.1)s = 5T \rightarrow T = (0.48 \pm 0.02)s$

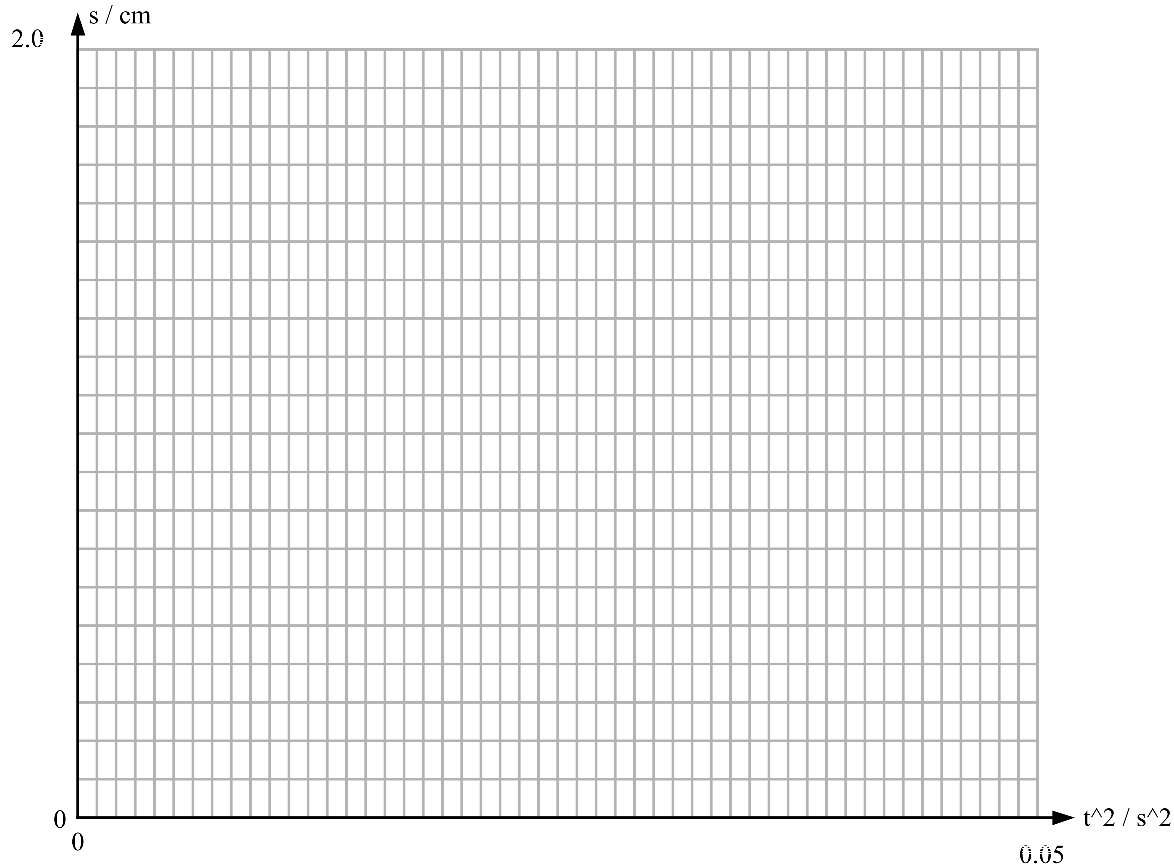
For **twenty** oscillations: $(9.4 \pm 0.1)s = 20T \rightarrow T = (0.047 \pm 0.005)s \approx (0.047 \pm 0.01)s$

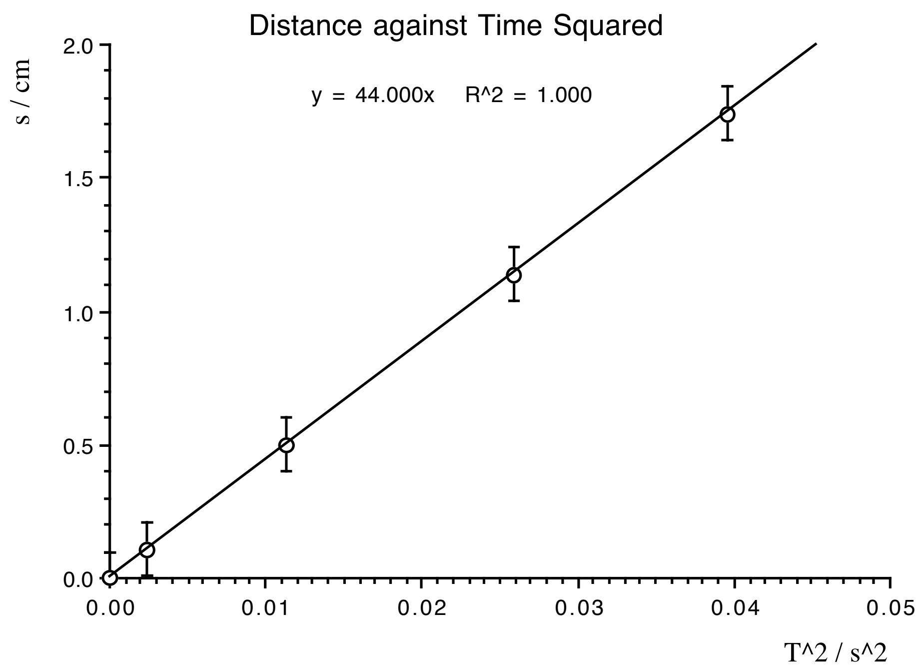
10. A computer interface is used to measure the position (s / cm) of an object under uniform acceleration $\left(a / cm s^{-2}\right)$ as a function of time (t). The uncertainty in the time measurement is very small, about $\Delta t = \pm 0.0001 s$, and so you can ignore it, while the uncertainty in the distance is significant, where $\Delta s = \pm 0.1 cm$. Here is the data.

Time t / s	t^2 / s^2	Distance s / cm
0.0000	No data point	0.0
0.0493	*0.002430*	0.1
0.1065	*0.011342*	0.5
0.1608	*0.025857*	1.1
0.1989	*0.039561*	1.7

The motion of the body can be described by the equation $s = (1/2) a t^2 = m t^2$ where the constant $m = 0.5 a$. A graph of distance against the square of time should have a slope of "m". Acceleration is determined by first finding the slope, where $a = 2 \times slope$. Use the above data and graph *distance* against *time squared*, construct uncertainty bars for the distance points, and then determine the best straight line and solve for the acceleration. Do not include the origin in your data points.

Distance against Time Squared





Acceleration: $a = 2 \times slope = 2 \times \left(44.0 cm\, s^{-2}\right) = 88.0\, cm\, s^{-2} \approx 88\, cm\, s^{-2}$

11. In an optical experiment a deflected ray of light is measured to be at an angle $\theta = (23 \pm 1)°$. Find the *sine* of this angle and then determine the *minimum* and *maximum* acceptable values of this experimental value.

    Next, express the experimental result in the form $\tan\theta \pm \Delta(\tan\theta)$.

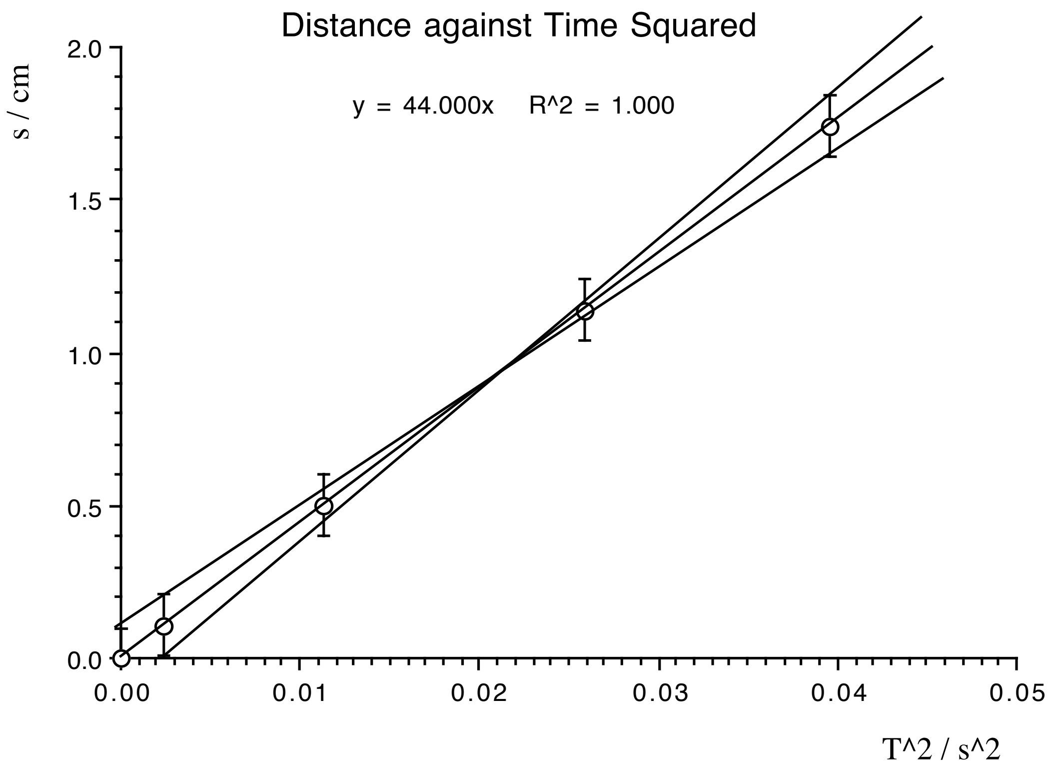
For the angle: $\sin 23° = 0.3907311 \approx 0.39$

For the maximum angle: $\sin(23+1)° = \sin 24° = 0.4067366 \approx 0.41$

For the minimum angle: $\sin(23-1)° = \sin 22° = 0.3746066 \approx 0.37$

$\tan\theta \pm \Delta(\tan\theta) = 0.39 \pm 0.2$

12. On your graph above (in problem 10) draw the minimum and maximum slopes on graph and determine the range of acceleration based on these two extreme slope values.



$$a_{\min} = 2 \times slope_{\min} = 2\left[\frac{(1.90 - 0.12)cm}{(0.045 - 0)s^2}\right] = 79.1 cm\, s^{-2}$$

$$a_{\max} = 2 \times slope_{\max} = 2\left[\frac{(1.90 - 0)cm}{(0.040 - 0.0025)s^2}\right] = 101.3 cm\, s^{-2}$$

Range is about $22.2 cm\, s^{-2}$ which is not symmetrical but close to being $\pm 11 cm\, s^{-2}$.

Hence we can say that the acceleration is $a \pm \Delta a \approx (88 \pm 11) cm\, s^{-2}$.

13. Density is the ratio of mass to volume, where $\rho = \frac{m}{V}$.

What is the *density* of a material if $m = 12.4\,kg$ and $V = 6.68\,m^3$?

Determine the least uncertainty in the mass and in the volume, and then calculate the uncertainty in the density value.

Express the uncertainty in the density ρ as an absolute value $\pm\Delta\rho$, as a fractional ratio $\pm\frac{\Delta\rho}{\rho}$, and as a percentage $\pm\Delta\rho\%$.

$$\rho = \frac{m}{V} = \frac{12.4kg}{6.68m^3} = 1.8562874\,kg\,m^{-3} \approx 1.86\,kg\,m^{-3} \text{ (to 3 SF)}$$

$$\text{Uncertainty in mass: } \Delta m = \pm 0.1\,kg \text{ or } \frac{0.1kg}{12.4kg} \times 100\% = 0.80645\% \approx 1\%$$

$$\text{Uncertainty in volume: } \Delta V = \pm 0.01\,m^3 \text{ or } \frac{0.01m^3}{6.68m^3} \times 100\% = 0.1497006\% \approx 0.1\%$$

Uncertainty in density is the sum of the uncertainty percentage of mass and volume, but the volume is one-tenth that of the mass, so we just keep the resultant uncertainty at 1%.

$$\rho = 1.86kg\,m^{-3} \pm 1\% \text{ (for a percentage of uncertainty)}$$

Where 1% of the density is $0.0186\,kg\,m^{-3}$ we can then write:

$$\rho = 1.86\,kg\,m^{-3} \pm 0.0186\,kg\,m^{-3} = \left(1.86 \pm \frac{0.02}{1.86}\right) kg\,m^{-3} \text{ for fractional uncertainty, and}$$

$$\rho = 1.86\,kg\,m^{-3} \pm 0.0186\,kg\,m^{-3} = (1.86 \pm 0.02) kg\,m^{-3} \text{ for absolute uncertainty}$$

14. What is the uncertainty in the calculated area of a circle whose radius is determined to be $r = (14.6 \pm 0.5)\,cm$?

$$\text{Area: } A = \pi r^2 \text{ where } r = (14.5 \pm 0.5) cm$$

$$\text{Percentage of uncertainty: } \frac{0.5cm}{14.6cm} \times 100\% = 3.42\%$$

The percentage is times 2 when squared = $2 \times 3.42\% = 6.849\%$

$$\text{Area: } A = \pi (14.6cm)^2 = 669.66189cm^2$$

Area uncertainty is about 7%, or in absolute terms, $6.849\% \times 699.6cm^2 = 45.86cm^2 \approx 46cm^2$

Therefore area: $A \pm \Delta A = (670. \pm 46)cm^2 \approx (6.7 \pm 0.5) \times 10^2\, cm^2$

15. An electrical resistor has a 2% tolerance and is marked $R = 1800\,\Omega$. What is the range of acceptable values that the resistor might have? An electrical current of $I = (2.1 \pm 0.1)\,mA$ flows through the resistor. What is the *uncertainty* in the calculated *voltage* across the resistor where the voltage is given as $V = I\,R$?

Resistance: $1800\,\Omega \times 2\% = \pm 36\,\Omega$ with a range from $1764\,\Omega$ to $1836\,\Omega$

Voltage: $V = I\,R$ where $\Delta R\% = 2\%$ and $\Delta I\% = \dfrac{0.1mA}{2.1mA} \times 100\% = 4.76\%$

The sum of the percentages is $2\% + 4.76\% = 6.76\% \approx 7\%$ and this is the uncertainty in the calculated voltage: $V = 7938mV = 7.938V$ at $6.76\% \times 7.928V = 0.5359V$

Therefore: $V \pm \Delta V = (7.9 \pm 0.5)V$

16. An accelerating object has an initial speed of $u = (12.4 \pm 0.1)\,m\,s^{-1}$ and a final speed of $v = (28.8 \pm 0.2)\,m\,s^{-1}$. The time interval for this change in speed is $\Delta t = (4.2 \pm 0.1)s$. Acceleration is defined as $a = (v - u)/\Delta t$. Calculate the *acceleration* and its *uncertainty*.

$$a = \frac{v - u}{t} = \frac{28.8m\,s^{-1} - 12.4m\,s^{-1}}{4.2s} = 3.90m\,s^{-2}$$

Uncertainty in numerator: $\Delta u + \Delta v = 0.1m\,s^{-1} + 0.2m\,s^{-1} = 0.3m\,s^{-1}$

As a percentage: $\dfrac{0.3m\,s^{-1}}{28.8m\,s^{-1} - 12.4m\,s^{-1}} \times 100\% = 1.829\%$

Percentage in time: $\dfrac{0.1s}{4.2s} \times 100\% = 2.380\%$

Percentage of uncertainty in acceleration is the sum: $\Delta a\% = 1.829\% + 2.380\% = 4.209\%$

Therefore acceleration: $3.90m\,s^{-2} \pm 4\% = (3.90 \pm 0.16)m\,s^{-2} \approx (3.9 \pm 0.2)m\,s^{-2}$

17. What are the *volume* and its *uncertainty* for a sphere with a radius of $r = (21 \pm 1)\,mm$?

Volume $V = \frac{4}{3}\pi r^3$ where $r = (21 \pm 1)mm$. As a percentage: $\frac{1mm}{21mm} \times 100\% = 4.76\%$

$$V = \frac{4\pi(21mm)^3}{3} = 38792.39\,mm^3$$

The cube on the radius made the 4.76% increase three times: $3 \times 4.76\% = 14.28\%$

The volume $= 38792.39cm^3 \pm 14.28\% \approx (38792.39 \pm 5539.55)cm^3$

In correct form, that would be: $V \pm \Delta V \approx (3900 \pm 6000)cm^3 = (3.9 \pm 0.6) \times 10^4\,cm^3$

As a percentage, $3.9 \times 10^4\,cm^3 \pm 14\%$

18. Frequency and period are related as reciprocals. What are the *period* and its *absolute uncertainty* when the frequency of $1.00 \times 10^3\,Hz$ is known to 2%?

$2\% \times 1000\,Hz = 20Hz$ and where $T = \frac{1}{f}$

Max. Period: $T_{\max} = \frac{1}{f_{\min}} = \frac{1}{f - \Delta f} = \frac{1}{1000.Hz - 20Hz} = \frac{1}{980Hz} = 0.001020408s$

Max. to 3 SF is thus $T_{\max} = 1.024 \times 10^{-3}s$

Min. Period: $T_{\min} = \frac{1}{f_{\max}} = \frac{1}{f + \Delta f} = \frac{1}{1000.Hz + 20Hz} = \frac{1}{1020Hz} = 0.000980392s$

Min. to 3 SF is thus $T_{\min} = 0.980 \times 10^{-3}s$

Uncertainty Above: $\left(1.02 \times 10^{-3}s\right) - \left(1.00 \times 10^{-3}s\right) = 0.02 \times 10^{-3}s$

Uncertainty Below: $\left(1.00 \times 10^{-3}s\right) - \left(0.98 \times 10^{-3}s\right) = 0.02 \times 10^{-3}s$

Period: $T \pm \Delta T = (1.00 \pm 0.02) \times 10^{-3}s$

—continued—

If we ignore the asymmetry here (which cancels out with SF) and just carry through the percentage, we would get:

$$T = \frac{1}{f} = \frac{1}{1000\,Hz} = 0.00100s \text{ or } 1.00 \times 10^{3} s$$

At 2% of this period time we get $\pm 0.00002\,s$ or $0.02 \times 10^{-3} s$

Therefore the period can be written as $T \pm \Delta T = (1.00 \pm 0.02)s$

19. Einstein's famous equation relates energy and mass with the square of the speed of light, where $E = mc^2$. What is the *percentage of uncertainty* and the *absolute uncertainty* of the *energy* for a mass $m = 1.00\,kg$ where the speed of light is $c = 3.00 \times 10^8\, m\,s^{-1}$?

$$E = mc^2 = (1.00kg)\left(3.00 \times 10^8\, m\,s^{-1}\right)^2 = 9.00 \times 10^{16}\, J$$

The minimum uncertainty in mass is the least significant digit: $\frac{0.01}{1.00} \times 100\% = 1\%$

The same for the speed of light: $\frac{0.01 \times 10^8\, m\,s^{-1}}{3.00 \times 10^8\, m\,s^{-1}} \times 100\% = \frac{0.01}{3.00} \times 100\% = 0.333\%$

We double the uncertainty percentage when squaring, hence $\Delta c^2 \% = 2 \times 0.333\% = 0.667\%$

The percentage of uncertainty in energy is then the sum of these: $1\% + 0.667\% = 1.667\%$

This is about 2% or in absolute terms, $\pm 1.5003 \times 10^{15}\, J$ or $\pm 0.15003 \times 10^{16}\, J$

Therefore: $E \pm \Delta E\% = 9.00 \times 10^{16}\, J \pm 2\%$

Therefore: $E \pm \Delta E = (9.00 \pm 0.15)J \approx (9.0 \pm 0.2)J$